Complex Geometry Exercises

Week 9

Exercise 1. *Let X, Y be closed complex manifolds. Prove that they are projective if and only their product is. Give two alternative proofs, using*

(i) the Serge embedding, and

(ii) the Kodaira embedding theorem.

Exercise 2. *Let $L \to \Sigma$ be a line bundle on a Riemann surface Σ. Prove the following assertions.*

(i) If $\deg(L) < 0$, then $H^0(\Sigma, L) = 0$.

(ii) If $\deg(L) > \deg(K_\Sigma)$, $H^1(\Sigma, L) = 0$.

(iii) If $\deg(L) > \deg(K_\Sigma) + 2$, its pluricanonical map ϕ_L is an embedding.

In particular, all Riemann surfaces are projective.

Exercise 3. *Prove that the blow-up of a projective manifold is again projective.*

Exercise 4. *Show that any vector bundle E on a projective manifold X can be written as a quotient $(L^k)^{\oplus l} \to E$ with L an ample line bundle, $k \ll 0$ and $l \gg 0$.*

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(continues on the back)

Exercise 6. *Prove that a complex torus* $\mathbb{T}_\Gamma = \mathbb{C}^n/\Gamma$ *is projective if and only if there exists an alternating bilinear form*

$$\omega : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$$

such that

- $\omega(iu, iv) = \omega(u, v)$,
- $\omega(\cdot, i\,\cdot)$ *is positive definite, and*
- $\omega(u, v) \in \mathbb{Z}$ *for all* $u, v \in \Gamma$

Exercise 7. *Let* X *be a compact Kähler manifold and*

$$Alb(X) := H^0(X, \Omega_X)^*/H_1(X, \mathbb{Z})$$

where

$$H_1(X, \mathbb{Z}) \to H^0(X, \Omega_X)^*, \quad \gamma \mapsto \left(\alpha \mapsto \int_\gamma \alpha\right).$$

Prove the following statements.

(i) $Alb(X)$ *is a complex torus.*

(ii) *Fixing a base point* $z_0 \in X$, *yields a holomorphic map* $X \to Alb(X)$.

(iii) *For* $\mathbb{T}_\Gamma = \mathbb{C}^n/\Gamma$, *the map* $\mathbb{T}_\Gamma \to Alb(\mathbb{T}_\Gamma)$ *is a biholomorphism.*